

On partition of unities generated by entire functions and Gabor frames in $L^2(\mathbb{R}^d)$ and $\ell^2(\mathbb{Z}^d)^*$

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Abstract

We characterize the entire functions P of d variables, $d \geq 2$, for which the \mathbb{Z}^d -translates of $P\chi_{[0,N]^d}$ satisfy the partition of unity for some $N \in \mathbb{N}$. In contrast to the one-dimensional case, these entire functions are not necessarily periodic. In the case where P is a trigonometric polynomial, we characterize the maximal smoothness of $P\chi_{[0,N]^d}$, as well as the function that achieves it. A number of especially attractive constructions are achieved, e.g., of trigonometric polynomials leading to any desired (finite) regularity for a fixed support size. As an application we obtain easy constructions of matrix-generated Gabor frames in $L^2(\mathbb{R}^d)$, with small support and high smoothness. By sampling this yields dual pairs of finite Gabor frames in $\ell^2(\mathbb{Z}^d)$.

Keywords Entire functions, trigonometric polynomials, partition of unity, dual frame pairs, Gabor systems, tight frames

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1 Introduction

Partition of unity conditions appear in many different contexts in analysis, e.g., within harmonic analysis [7, 8]. In this paper we characterize the entire functions $P : \mathbb{C}^d \rightarrow \mathbb{C}$, $d \geq 2$, which, for some fixed $N \in \mathbb{N}$, satisfy the partition of unity condition

$$\sum_{\mathbf{n} \in \mathbb{Z}^d} P(\mathbf{x} + \mathbf{n}) \chi_{[0,N]^d}(\mathbf{x} + \mathbf{n}) = 1, \quad \forall \mathbf{x} \in \mathbb{R}^d. \quad (1.1)$$

In contrast to the case $d = 1$ treated in [2] such a function P is not necessarily $(N\mathbb{Z})^d$ -periodic. In the special case of periodic entire functions P we derive an alternative and more direct characterization of the partition of unity condition in terms of the Fourier coefficients

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of P . For the case where P is a trigonometric polynomial the maximal smoothness of $P\chi_{[0,N]^d}$ is characterized, as well as the entire functions P that attain it.

The approach leads to a number of explicit constructions of functions P that yield a partition of unity and have desired smoothness. We apply these results to provide very easy constructions of pairs of dual Gabor frames with a number of attractive properties, such as small support and high regularity. Compared to the B-spline based frame constructions in [3] these constructions are considerably more convenient: we avoid a complicated “book keeping,” and the dual window has the same support as the window itself. Due to the compact support and continuity of the windows the results lead to an easy way to construct finite dual Gabor frames in $\ell^2(\mathbb{Z}^d)$ as well. For more information on Gabor frames we refer to the monographs [9, 1].

The paper is organized as follows. In Section 2 we characterize the entire functions that have the partition of unity property, as well in the general case as in the periodic case. The regularity issue is considered in Section 3, and the applications to Gabor frames are in Sections 4 and 5.

2 Partition of unity for entire functions

Our first goal is to characterize the entire functions $P : \mathbb{C}^d \rightarrow \mathbb{C}$, $d \geq 2$, which, for some fixed $N \in \mathbb{N}$, satisfy the partition of unity condition (1.1). In order to do this we need to introduce some notation. For $\mathbf{y} \in \mathbb{C}^{d-1}$, write $\mathbf{y} = (y_1, \dots, y_{d-1})$. For any $j \in \{1, \dots, d\}$, define the function

$$P_j : \mathbb{C} \times \mathbb{C}^{d-1} \rightarrow \mathbb{C}, P_j(x, \mathbf{y}) := P(y_1, \dots, y_{j-1}, x, y_j, \dots, y_{d-1}).$$

The function P_j is introduced in order to have a notation that allows us to “pull out” the j th variable; in fact, for any $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{C}^d$, letting $\mathbf{y} := (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_d) \in \mathbb{C}^{d-1}$ yields that

$$P(\mathbf{x}) = P(x_1, \dots, x_d) = P_j(x_j, (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_d)) = P_j(x_j, \mathbf{y}).$$

In particular, $P(\mathbf{x}) = P(x_1, \dots, x_d) = P_1(x_1, (x_2, \dots, x_d))$. Given $N \in \mathbb{N}$, the symbol $\sum_{\mathbf{n} \in \mathbb{Z}_N^{d-1}}$ will denote a sum over the $\mathbf{n} = (n_1, \dots, n_{d-1}) \in \mathbb{Z}^{d-1}$ for which all coordinates are between 0 and $N - 1$, i.e.,

$$\sum_{\mathbf{n} \in \mathbb{Z}_N^{d-1}} := \sum_{n_1=0}^{N-1} \cdots \sum_{n_{d-1}=0}^{N-1}.$$

Now, for any $j \in \{1, \dots, d\}$, define the function

$$Q_j : \mathbb{C} \times \mathbb{C}^{d-1} \rightarrow \mathbb{C}, Q_j(x, \mathbf{y}) := \sum_{\mathbf{n} \in \mathbb{Z}_N^{d-1}} P_j(x, \mathbf{y} + \mathbf{n}).$$

Explicitly,

$$Q_j(x, y_1, \dots, y_{d-1}) = \sum_{n_1=0}^{N-1} \cdots \sum_{n_{d-1}=0}^{N-1} P(y_1 + n_1, \dots, y_{j-1} + n_{j-1}, x, y_j + n_j, \dots, y_{d-1} + n_{d-1}).$$

We will now characterize the entire functions P satisfying the partition of unity condition in terms of (any of) the associated functions Q_j .

Proposition 2.1 *Let $P : \mathbb{C}^d \rightarrow \mathbb{C}$ be an entire function. Let $N \in \mathbb{N}$, and consider any $j \in \{1, \dots, d\}$. Then the following are equivalent:*

- (a) P satisfies the partition of unity condition (1.1);
- (b) For any $\mathbf{y} \in [0, 1]^{d-1}$, the restriction of $Q_j(\cdot, \mathbf{y})$ to \mathbb{R} is N -periodic and the Fourier coefficients $c_k(\mathbf{y})$ in the expansion

$$Q_j(x, \mathbf{y}) = \sum_{k \in \mathbb{Z}} c_k(\mathbf{y}) e^{2\pi i k x / N}, \quad x \in \mathbb{R} \quad (2.1)$$

satisfy that $c_k(\mathbf{y}) = \frac{1}{N} \delta_{k,0}$ for $k \in N\mathbb{Z}$.

Proof. (a) \Rightarrow (b) Assume first that (1.1) holds, and take any $j \in \{1, \dots, d\}$. Then $\sum_{\mathbf{n} \in \mathbb{Z}_N^d} P(\mathbf{x} + \mathbf{n}) = 1$, $\forall \mathbf{x} \in [0, 1]^d$. Thus, for any fixed $\mathbf{y} \in [0, 1]^{d-1}$,

$$\sum_{\ell=0}^{N-1} Q_j(x + \ell, \mathbf{y}) = \sum_{\ell=0}^{N-1} \sum_{\mathbf{n} \in \mathbb{Z}_N^{d-1}} P_j(x + \ell, \mathbf{y} + \mathbf{n}) = 1, \quad x \in [0, 1]. \quad (2.2)$$

Since $Q_j(\cdot, \mathbf{y})$ is an entire function, (2.2) then holds for all $x \in \mathbb{R}$. Replacing x by $x + 1$ in (2.2) and subtracting the two expressions shows that $Q_j(x + N, \mathbf{y}) = Q_j(x, \mathbf{y})$, $x \in \mathbb{R}$. So we conclude that the restriction of $Q_j(\cdot, \mathbf{y})$ to \mathbb{R} is N -periodic. Writing $Q_j(\cdot, \mathbf{y})$ as the Fourier series (2.1), the equation (2.2) takes the form

$$\sum_{k \in \mathbb{Z}} c_k(\mathbf{y}) \left[1 + e^{2\pi i k / N} + \cdots + (e^{2\pi i k / N})^{N-1} \right] e^{2\pi i k x / N} = 1. \quad (2.3)$$

We note that

$$1 + e^{2\pi i k / N} + \cdots + (e^{2\pi i k / N})^{N-1} = \begin{cases} N, & k \in N\mathbb{Z} \\ 0, & k \notin N\mathbb{Z}. \end{cases} \quad (2.4)$$

From (2.3) and (2.4), we see that $c_k(\mathbf{y}) = \frac{1}{N} \delta_{k,0}$ for $k \in N\mathbb{Z}$, as claimed.

(a) \Leftarrow (b) Let again $j \in \{1, \dots, d\}$, and consider any $\mathbf{x} \in [0, 1]^d$. Then

$$\begin{aligned} \sum_{\mathbf{n} \in \mathbb{Z}^d} P(\mathbf{x} + \mathbf{n}) \chi_{[0, N]^d}(\mathbf{x} + \mathbf{n}) &= \sum_{\mathbf{n} \in \mathbb{Z}_N^d} P(\mathbf{x} + \mathbf{n}) \\ &= \sum_{n_1=0}^{N-1} \sum_{n_2=0}^{N-1} \cdots \sum_{n_d=0}^{N-1} P_j(x_j + n_j, (x_1 + n_1, \dots, x_{j-1} + n_{j-1}, x_{j+1} + n_{j+1}, \dots, x_d + n_d)). \end{aligned}$$

With $\mathbf{y} := (x_1 + n_1, \dots, x_{j-1} + n_{j-1}, x_{j+1} + n_{j+1}, \dots, x_d + n_d)$, the assumption in (b) and (2.4) now yields that

$$\begin{aligned} \sum_{\mathbf{n} \in \mathbb{Z}^d} P(\mathbf{x} + \mathbf{n}) \chi_{[0, N]^d}(\mathbf{x} + \mathbf{n}) &= \sum_{n_j=0}^{N-1} Q_j(x_j + n_j, \mathbf{y}) = \sum_{n_j=0}^{N-1} \sum_{k \in \mathbb{Z}} c_k(\mathbf{y}) e^{2\pi i k(x_j + n_j)/N} \\ &= \sum_{k \in \mathbb{Z}} c_k(\mathbf{y}) \left[1 + e^{2\pi i k/N} + \dots + (e^{2\pi i k/N})^{N-1} \right] e^{2\pi i k x_j/N} = 1. \end{aligned}$$

By periodicity of $\sum_{\mathbf{n} \in \mathbb{Z}^d} P(\cdot + \mathbf{n}) \chi_{[0, N]^d}(\cdot + \mathbf{n})$, (1.1) therefore holds for all $\mathbf{x} \in \mathbb{R}^d$. \square

Note that if the conditions in Proposition 2.1 hold, then (b) actually holds for all $\mathbf{y} \in \mathbb{R}^d$. From the proof we also see immediately that a similar result holds with the “square” $[0, N]^d$ replaced by a rectangle $[0, N_1] \times \dots \times [0, N_d]$, where $N_1, \dots, N_d \in \mathbb{N}$.

In [2] it was proved that in the case $d = 1$, an entire function $P : \mathbb{C} \rightarrow \mathbb{C}$ satisfying (1.1) is automatically N -periodic. The following example shows that this does not generalize to the case $d > 1$. In fact, an entire function $P : \mathbb{C}^d \rightarrow \mathbb{C}$ satisfying (1.1) might not be periodic in any of the variables:

Example 2.2 Let $d = 2$ and let $f : \mathbb{C} \rightarrow \mathbb{C}$ denote an entire function. Consider the entire function $P : \mathbb{C}^2 \rightarrow \mathbb{C}$ given by

$$P(x_1, x_2) := e^{\pi i x_2} f(x_1) + \frac{1}{4} + e^{\pi i x_1} f(x_2).$$

An easy direct computation shows that P satisfies the partition of unity condition (1.1) for $N = 2$. Alternatively, let $j = 1$ and fix $x_2 \in [0, 1]$. Then

$$Q_1(x_1, x_2) = P(x_1, x_2) + P(x_1, x_2 + 1) = \frac{1}{2} + e^{\pi i x_1} (f(x_2) + f(x_2 + 1)).$$

Thus, for any fixed x_2 , $Q_1(\cdot, x_2)$ is 2-periodic and satisfies the condition (b) in Proposition 2.1; this again implies that P satisfies the partition of unity condition (1.1). However, in general $P(\cdot, \cdot)$ is periodic neither in the first variable nor in the second variable. \square

In order to have an extra technical tool at our disposal (namely, Fourier series) we will now restrict our attention to entire functions $P : \mathbb{C}^d \rightarrow \mathbb{C}$ that are $(N\mathbb{Z})^d$ -periodic, i.e., entire functions P for which the restriction to \mathbb{R}^d can be written in the form

$$P(\mathbf{x}) := \sum_{\mathbf{k} \in \mathbb{Z}^d} c_{\mathbf{k}} e^{2\pi i \mathbf{k} \cdot \mathbf{x}/N}, \quad \mathbf{x} \in \mathbb{R}^d. \quad (2.5)$$

For this class of entire functions we will now give a convenient characterization of the functions having the partition of unity property in terms of the Fourier coefficients $c_{\mathbf{k}}$. The reader who checks the proof will notice that a similar result holds for entire functions that are periodic along a lattice $N_1\mathbb{Z} \times \dots \times N_d\mathbb{Z}$ for some $N_1, \dots, N_d \in \mathbb{N}$.

Corollary 2.3 *An entire $(N\mathbb{Z})^d$ -periodic function P of the form (2.5) satisfies (1.1) if and only if*

$$c_{\mathbf{k}} = \frac{1}{N^d} \delta_{\mathbf{k}, \mathbf{0}}, \quad \forall \mathbf{k} \in (N\mathbb{Z})^d, \quad (2.6)$$

Proof. We will apply Proposition 2.1 for the choice $j = 1$. For $x \in \mathbb{R}$ and $\mathbf{y} \in [0, 1]^{d-1}$, we have

$$Q_1(x, \mathbf{y}) = \sum_{\mathbf{n} \in \mathbb{Z}_N^{d-1}} P_1(x, \mathbf{y} + \mathbf{n}) = \sum_{\mathbf{n} \in \mathbb{Z}_N^{d-1}} P(x, \mathbf{y} + \mathbf{n}). \quad (2.7)$$

Let us write the Fourier series (2.5) in a slightly different form, namely, as

$$P(\mathbf{x}) = P(x_1, \tilde{\mathbf{x}}) = \sum_{k \in \mathbb{Z}} \sum_{\mathbf{m} \in \mathbb{Z}^{d-1}} c_{k, \mathbf{m}} e^{2\pi i k x_1 / N} e^{2\pi i \mathbf{m} \cdot \tilde{\mathbf{x}} / N},$$

where $\tilde{\mathbf{x}} := (x_2, \dots, x_d)$. Inserting this in (2.7) yields

$$\begin{aligned} Q_1(x, \mathbf{y}) &= \sum_{\mathbf{n} \in \mathbb{Z}_N^{d-1}} \sum_{k \in \mathbb{Z}} \sum_{\mathbf{m} \in \mathbb{Z}^{d-1}} c_{k, \mathbf{m}} e^{2\pi i k x / N} e^{2\pi i \mathbf{m} \cdot (\mathbf{y} + \mathbf{n}) / N} \\ &= \sum_{k \in \mathbb{Z}} \left(\sum_{\mathbf{n} \in \mathbb{Z}_N^{d-1}} \sum_{\mathbf{m} \in \mathbb{Z}^{d-1}} c_{k, \mathbf{m}} e^{2\pi i \mathbf{m} \cdot (\mathbf{y} + \mathbf{n}) / N} \right) e^{2\pi i k x / N} = \sum_{k \in \mathbb{Z}} c_k(\mathbf{y}) e^{2\pi i k x / N}, \end{aligned}$$

where the coefficients $c_k(\mathbf{y})$ are given by

$$c_k(\mathbf{y}) = \sum_{\mathbf{n} \in \mathbb{Z}_N^{d-1}} \sum_{\mathbf{m} \in \mathbb{Z}^{d-1}} c_{k, \mathbf{m}} e^{2\pi i \mathbf{m} \cdot (\mathbf{y} + \mathbf{n}) / N} = \sum_{\mathbf{m} \in \mathbb{Z}^{d-1}} c_{k, \mathbf{m}} e^{2\pi i \mathbf{m} \cdot \mathbf{y} / N} \sum_{\mathbf{n} \in \mathbb{Z}_N^{d-1}} e^{2\pi i \mathbf{m} \cdot \mathbf{n} / N}.$$

The sum $\sum_{\mathbf{n} \in \mathbb{Z}_N^{d-1}} e^{2\pi i \mathbf{m} \cdot \mathbf{n} / N}$ is only nonzero whenever $\mathbf{m} = N\mathbf{p}$ for some $\mathbf{p} \in \mathbb{Z}^{d-1}$, in which case the sum is N^{d-1} ; inserting this yields

$$c_k(\mathbf{y}) = N^{d-1} \sum_{\mathbf{p} \in \mathbb{Z}^{d-1}} c_{k, N\mathbf{p}} e^{2\pi i \mathbf{p} \cdot \mathbf{y}}. \quad (2.8)$$

The condition (b) in Proposition 2.1, namely, that the coefficients $c_k(\mathbf{y})$ in (2.8) satisfy that $c_k(\mathbf{y}) = \frac{1}{N} \delta_{k, 0}$ for $k \in N\mathbb{Z}$, is equivalent to

$$c_{0, N\mathbf{p}} = \frac{1}{N^d} \delta_{\mathbf{0}, \mathbf{p}}, \quad \text{for } \mathbf{p} \in \mathbb{Z}^{d-1}; \quad c_{k, N\mathbf{p}} = 0, \quad \text{for } k \in N\mathbb{Z} \setminus \{0\}, \mathbf{p} \in \mathbb{Z}^{d-1},$$

which again is equivalent with $c_{\mathbf{k}} = \frac{1}{N^d} \delta_{\mathbf{k}, \mathbf{0}}, \forall \mathbf{k} \in (N\mathbb{Z})^d$. This completes the proof. \square

Note that by Corollary 2.3 the entire $(N\mathbb{Z})^d$ -periodic functions P in the form (2.5) which satisfies (1.1), precisely are the ones having the form $P(\mathbf{x}) = \frac{1}{N^d} + \sum_{\mathbf{k} \in \mathbb{Z}^d \setminus (N\mathbb{Z})^d} c_{\mathbf{k}} e^{2\pi i \mathbf{k} \cdot \mathbf{x} / N}$. In general such functions are not tensor products of d one-dimensional Fourier series.

Our next goal is to construct entire functions P such that $P\chi_{[0, N]^d}$ satisfies the partition of unity condition and has desired regularity. Clearly, already the continuity of $P\chi_{[0, N]^d}$ forces P to vanish along the boundary of the “square” $[0, N]^d$. The following example shows that even this extra constraint does not imply that P is periodic:

Example 2.4 Let $d = 2$ and $N = 2$. Our purpose is to construct a nonperiodic entire function which satisfies the partition of unity condition and vanishes on the boundary of $[0, 2]^2$. Let f, g be any entire functions. A calculation like in Example 2.2 shows that the entire function

$$P(x_1, x_2) := \frac{1}{4} + e^{\pi i x_2} f(x_1) + e^{\pi i x_1} f(x_2) + e^{-\pi i x_2} g(x_1) + e^{-\pi i x_1} g(x_2)$$

satisfies the partition of unity condition. Now choose f as an entire *nonperiodic* function such that $f(0) = f(2) = 0$, and let $g(x) := -\frac{1}{4} - f(x) + \frac{1}{8}e^{-\pi i x}$. Then $g(0) = g(2) = -1/8$. Also, for any $x_1, x_2 \in \mathbb{R}$, $P(x_1, 0) = P(x_1, 2) = P(0, x_2) = P(2, x_2) = 0$. Thus, the function P vanishes on the boundary of $[0, 2]^2$. Note that

$$\begin{aligned} P(x_1, x_2) &= \frac{1}{4} + e^{\pi i x_2} f(x_1) + e^{\pi i x_1} f(x_2) + e^{-\pi i x_2} \left(-\frac{1}{4} - f(x_1) + \frac{1}{8}e^{-\pi i x_1} \right) \\ &\quad + e^{-\pi i x_1} \left(-\frac{1}{4} - f(x_2) + \frac{1}{8}e^{-\pi i x_2} \right) \\ &= \frac{1}{4} + f(x_1) [e^{\pi i x_2} - e^{-\pi i x_2}] + f(x_2) [e^{\pi i x_1} - e^{-\pi i x_1}] \\ &\quad - \frac{1}{4}e^{-\pi i x_1} - \frac{1}{4}e^{-\pi i x_2} + \frac{1}{4}e^{-\pi i x_1}e^{-\pi i x_2}, \end{aligned}$$

which, by the assumptions on f , clearly is nonperiodic. \square

The Fourier series turn out to be the key ingredient in the regularity discussion in the next section, so we will continue to assume that P is periodic.

3 Regularity

Among the $(N\mathbb{Z})^d$ -periodic entire functions $P : \mathbb{C}^d \rightarrow \mathbb{C}$ satisfying the partition of unity condition, we will now restrict our attention to the ones with only a finite number of nonzero Fourier coefficients, i.e., the trigonometric polynomials. For such functions, the following result characterizes the maximal smoothness of $P\chi_{[0, N]^d}$, as well as the entire functions P that achieve it. Recall that for $L \in \mathbb{N}$, the space $C^{L-1}(\mathbb{R}^d)$ consists of the functions $f : \mathbb{R}^d \rightarrow \mathbb{C}$ for which all the partial derivatives

$$\frac{\partial^\ell f}{\partial x_1^{\ell_1} \partial x_2^{\ell_2} \cdots \partial x_d^{\ell_d}}, \quad 0 \leq \ell = \ell_1 + \cdots + \ell_d \leq L - 1,$$

exist and are continuous.

Theorem 3.1 *Let $K, N \in \mathbb{N}$. Assume that*

$$P(\mathbf{x}) = \sum_{\mathbf{k} \in (\mathbb{Z} \cap [-K, K])^d} c_{\mathbf{k}} e^{2\pi i \mathbf{k} \cdot \mathbf{x} / N}, \quad \mathbf{x} \in \mathbb{R}^d \quad (3.1)$$

is a real-valued trigonometric polynomial. Then the following hold:

(a) *There does not exist P of the form (3.1) such that $P\chi_{[0,N]^d} \in C^{2K}(\mathbb{R}^d)$;*

(b) *Fix $L \in \{1, 2, \dots, 2K\}$. Then $P\chi_{[0,N]^d} \in C^{L-1}(\mathbb{R}^d)$ if and only if*

$$P(\mathbf{x}) = P(x_1, \dots, x_d) = \prod_{j=1}^d \left(e^{\pi i x_j / N} \sin(\pi x_j / N) \right)^L A_L(\mathbf{x}) \quad (3.2)$$

for a trigonometric polynomial

$$A_L(\mathbf{x}) = \sum_{\mathbf{k} \in (\mathbb{Z} \cap [-K, K-L])^d} a_{\mathbf{k}} e^{2\pi i \mathbf{k} \cdot \mathbf{x} / N}; \quad (3.3)$$

(c) *Assume that P has the form (3.2). Then $P\chi_{[0,N]^d}$ satisfies the partition of unity condition (1.1) if and only if for all $\mathbf{m} \in (N\mathbb{Z})^d$,*

$$\sum_{\substack{\mathbf{j} + \mathbf{k} = \mathbf{m} \\ \mathbf{j} \in (\mathbb{Z} \cap [0, L])^d \\ \mathbf{k} \in (\mathbb{Z} \cap [-K, K-L])^d}} (-1)^{Ld - (j_1 + \dots + j_d)} \binom{L}{j_1} \dots \binom{L}{j_d} a_{\mathbf{k}} = \left(\frac{(2i)^L}{N} \right)^d \delta_{\mathbf{m}, \mathbf{0}}. \quad (3.4)$$

Proof. We will first prove (b). First note that if $P\chi_{[0,N]^d} \in C^{L-1}(\mathbb{R}^d)$, then for $j = 1, \dots, d$, we have $\frac{\partial^\ell P}{\partial x_j^\ell}(x_1, \dots, x_{j-1}, 0, x_{j+1}, \dots, x_d) = 0$, $\ell = 0, \dots, L-1$. Let us write (3.1) as $P(\mathbf{x}) = P(x_1, \dots, x_d) = \sum_{k_1=-K}^K c_{k_1}(x_2, \dots, x_d) e^{2\pi i k_1 x_1 / N}$, where

$$c_{k_1}(x_2, \dots, x_d) = \sum_{k_2=-K}^K \dots \sum_{k_d=-K}^K c_{\mathbf{k}} e^{2\pi i (k_2, \dots, k_d) \cdot (x_2, \dots, x_d) / N}.$$

By [2, Theorem 3.1], we know that whenever $L \leq 2K$, then $\frac{\partial^\ell P}{\partial x_1^\ell}(0, x_2, \dots, x_d) = 0$ for $\ell = 0, \dots, L-1$ if and only if

$$P(\mathbf{x}) = \left(e^{\pi i x_1 / N} \sin(\pi x_1 / N) \right)^L A_{1,L}(\mathbf{x}) \quad (3.5)$$

for a trigonometric polynomial

$$A_{1,L}(\mathbf{x}) := \sum_{k_1=-K}^{K-L} a_{k_1}(x_2, \dots, x_d) e^{2\pi i k_1 x_1 / N}. \quad (3.6)$$

We note that for some coefficients $d_{\mathbf{k}}$,

$$\begin{aligned} a_{k_1}(x_2, \dots, x_d) &= \sum_{k_2=-K}^K \dots \sum_{k_d=-K}^K d_{\mathbf{k}} e^{2\pi i (k_2, \dots, k_d) \cdot (x_2, \dots, x_d)} \\ &= \sum_{k_2=-K}^K \left(\sum_{k_3=-K}^K \dots \sum_{k_d=-K}^K d_{\mathbf{k}} e^{2\pi i (k_3, \dots, k_d) \cdot (x_3, \dots, x_d)} \right) e^{2\pi i k_2 x_2}. \end{aligned} \quad (3.7)$$

Now, by (3.5) and (3.6), $\frac{\partial^\ell P}{\partial x_2^\ell}(x_1, 0, x_3, \dots, x_d) = 0$ for $0 \leq \ell \leq L-1$ if and only if $\frac{\partial^\ell A_{1,L}}{\partial x_2^\ell}(x_1, 0, x_3, \dots, x_d) = 0$ for $0 \leq \ell \leq L-1$; or, if and only if $\frac{\partial^\ell a_{k_1}}{\partial x_2^\ell}(0, x_3, \dots, x_d) = 0$ for $0 \leq \ell \leq L-1$. By [2, Theorem 3.1] and (3.7) again, this is equivalent to

$$a_{k_1}(x_2, \dots, x_d) = \left(e^{\pi i x_2 / N} \sin(\pi x_2 / N) \right)^L \sum_{k_2=-K}^{K-L} a_{k_1, k_2}(x_3, \dots, x_d) e^{2\pi i k_2 x_2 / N}.$$

That is, $P(\mathbf{x}) = \prod_{j=1}^2 \left(e^{\pi i x_j / N} \sin(\pi x_j / N) \right)^L A_{2,L}(\mathbf{x})$ for a trigonometric polynomial

$$A_{2,L}(\mathbf{x}) = \sum_{k_1=-K}^{K-L} \sum_{k_2=-K}^{K-L} a_{k_1, k_2}(x_3, \dots, x_d) e^{2\pi i (k_1 x_1 + k_2 x_2) / N}.$$

Inductively, we have $P(\mathbf{x}) = \prod_{j=1}^d \left(e^{\pi i x_j / N} \sin(\pi x_j / N) \right)^L A_L(\mathbf{x})$ for a trigonometric polynomial

$$A_L(\mathbf{x}) = \sum_{k_1=-K}^{K-L} \dots \sum_{k_d=-K}^{K-L} a_{k_1, \dots, k_d} e^{2\pi i (k_1, \dots, k_d) \cdot (x_1, \dots, x_d) / N}.$$

Conversely, assume that $P(\mathbf{x}) = \prod_{j=1}^d \left(e^{\pi i x_j / N} \sin(\pi x_j / N) \right)^L A_L(\mathbf{x})$ for a trigonometric polynomial A_L of the form (3.3). Then for $0 \leq \ell_1 + \dots + \ell_d \leq L-1$, and for $1 \leq m \leq d$,

$$\frac{\partial^\ell P}{\partial x_1^{\ell_1} \partial x_2^{\ell_2} \dots \partial x_d^{\ell_d}}(x_1, \dots, x_{m-1}, 0, x_{m+1}, \dots, x_d) = 0$$

is trivially satisfied. Since P is entire and $(N\mathbb{Z})^d$ -periodic, $P_{\chi_{[0,N]^d}} \in C^{L-1}(\mathbb{R}^d)$. Hence (b) holds.

In order to prove (a), assume that $P_{\chi_{[0,N]^d}} \in C^{2K}(\mathbb{R}^d)$. Using (b) with $L = 2K$, we see that $P(\mathbf{x}) = \alpha \prod_{j=1}^d (\sin(\pi x_j / N))^{2K}$ for some $\alpha \in \mathbb{C}$. A direct calculation shows that for $(x_2, \dots, x_d) \in [0, N]^{d-1}$,

$$\frac{\partial^{2K} P}{\partial x_1^{2K}}(0, x_2, \dots, x_d) = \alpha \left(\frac{\pi}{N} \right)^{2K} (2K)! \prod_{j=2}^d (\sin(\pi x_j / N))^{2K} \neq 0.$$

This is a contradiction. Thus $P_{\chi_{[0,N]^d}} \notin C^{2K}(\mathbb{R}^d)$ and (a) holds.

For the proof of (c), we use the identity

$$\left(e^{\pi i x / N} \sin(\pi x / N) \right)^L = \left(\frac{e^{2\pi i x / N} - 1}{2i} \right)^L = \left(\frac{1}{2i} \right)^L \sum_{j=0}^L \binom{L}{j} e^{2\pi i j x / N} (-1)^{L-j};$$

then

$$\begin{aligned}
P(\mathbf{x}) &= \left(\frac{1}{2i}\right)^{Ld} \sum_{\mathbf{j} \in (\mathbb{Z} \cap [0, L])^d} \binom{L}{j_1} \cdots \binom{L}{j_d} e^{2\pi i \mathbf{j} \cdot \mathbf{x}/N} (-1)^{Ld - (j_1 + \cdots + j_d)} \sum_{\mathbf{k} \in (\mathbb{Z} \cap [-K, K-L])^d} a_{\mathbf{k}} e^{2\pi i \mathbf{k} \cdot \mathbf{x}/N} \\
&= \left(\frac{1}{2i}\right)^{Ld} \sum_{\mathbf{m} \in \mathbb{Z}^d} \left(\sum_{\substack{\mathbf{j} + \mathbf{k} = \mathbf{m} \\ \mathbf{j} \in (\mathbb{Z} \cap [0, L])^d \\ \mathbf{k} \in (\mathbb{Z} \cap [-K, K-L])^d}} (-1)^{Ld - (j_1 + \cdots + j_d)} \binom{L}{j_1} \cdots \binom{L}{j_d} a_{\mathbf{k}} \right) e^{2\pi i \mathbf{m} \cdot \mathbf{x}/N}.
\end{aligned}$$

By Corollary 2.3, the condition that $P_{\chi_{[0, N]^d}}$ satisfies the partition of unity condition is equivalent to (3.4). Hence (c) holds. \square

Let us use Theorem 3.1 to construct a partition of unity explicitly.

Example 3.2 Let $L = N = K = d = 2$. We will find P of the form (3.2) such that $P_{\chi_{[0, 2]^2}} \in C^1(\mathbb{R}^2)$ and the partition of unity condition (1.1) holds. Let $A_2(x_1, x_2) = \sum_{k_1, k_2 \in \{-2, -1, 0\}} a_{k_1, k_2} e^{\pi i (k_1 x_1 + k_2 x_2)}$, and assume that the coefficients a_{k_1, k_2} are real numbers and satisfy that

$$a_{k_1, k_2} = a_{-k_1-2, -k_2-2}, \quad k_1, k_2 = -2, -1, 0.$$

We now apply Theorem 3.1 (c). Due to the limitations on the summation indices \mathbf{j}, \mathbf{k} in (3.4), it is now enough to choose the coefficients a_{k_1, k_2} such that (3.4) is satisfied for $\mathbf{m} = (0, 0), \pm(2, 0), \pm(0, 2), \pm(2, 2)$. A direct calculation shows that these equations amount to the four equations $2a_{0,0} - 4a_{-1,0} + 2a_{0,-2} - 4a_{0,-1} + 4a_{-1,-1} = 4$; $a_{0,-2} - 2a_{0,-1} + a_{0,0} = 0$; $a_{0,-2} - 2a_{-1,0} + a_{0,0} = 0$; $a_{0,0} = 0$. If we assign a parameter $a_{0,-2} = t$, $t \in \mathbb{R}$, the solution can be written in vector form as

$$\begin{pmatrix} a_{0,0} \\ a_{-1,0} \\ a_{-1,-1} \\ a_{0,-1} \\ a_{0,-2} \end{pmatrix} = \begin{pmatrix} a_{-2,-2} \\ a_{-1,-2} \\ a_{-1,-1} \\ a_{-2,-1} \\ a_{-2,0} \end{pmatrix} = t \begin{pmatrix} 0 \\ 1/2 \\ 1/2 \\ 1/2 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

By direct calculation using (3.2) it now follows that for any $t \in \mathbb{R}$, the trigonometric polynomial

$$P(x_1, x_2) = \sin^2(\pi x_1/N) \sin^2(\pi x_2/N) [t(\cos(\pi x_1) + \cos(\pi x_2) + 2\cos(\pi(x_1 - x_2))) + 1/2] + 1]$$

satisfies the requirements. \square

Let us comment on the parameters K, N, L appearing in Theorem 3.1. Not surprisingly, Theorem 3.1 shows that the “budget of nonzero Fourier coefficients” for the entire function P in (3.1) limits the possible smoothness of $P_{\chi_{[0, N]^d}}$. If we fix $N \in \mathbb{N}$, e.g., if we want a

certain support size for the function $P\chi_{[0,N]^d}$, the condition (2.6) is automatically satisfied for $\mathbf{k} \neq \mathbf{0}$ if we take $K \leq N-1$ in (3.1). Thus, the characterization in Theorem 3.1 (b) yields an easy way to construct partition of unities with smoothness at most $2(N-1)-1 = 2N-3$. We note that if we for some $N \in \mathbb{N}$ want maximal smoothness of $P\chi_{[0,N]^d}$, there is a unique function P among the functions in (3.1) with $K \leq N-1$ that achieves it. In fact, fixing $N \in \mathbb{N}$ and taking $K = N-1, L = 2K$, yields that $K-L = -K$. Thus P must have the form

$$P(\mathbf{x}) = \prod_{j=1}^d (e^{\pi i x_j / N} \sin(\pi x_j / N))^{2K} \sum_{\mathbf{k} \in (\mathbb{Z} \cap [-K, -K])^d} a_{\mathbf{k}} e^{2\pi i \mathbf{k} \cdot \mathbf{x} / N};$$

that is, for some $a \in \mathbb{R}$,

$$P(\mathbf{x}) = a \prod_{j=1}^d e^{2\pi i K x_j / N} \sin^{2K}(\pi x_j / N) e^{-2\pi i (Kx_1 + \dots + Kx_d) / N} = a \prod_{j=1}^d \sin^{2N-2}(\pi x_j / N). \quad (3.8)$$

This construction stands out as the optimal one with regard to smoothness, and also as the simplest and most elegant one. Let us formulate the result formally:

Corollary 3.3 *Let $N \in \mathbb{N}$ and let*

$$P(\mathbf{x}) := \left(\frac{4^{N-1}}{N \binom{2N-2}{N-1}} \right)^d \prod_{j=1}^d \sin^{2N-2}(\pi x_j / N). \quad (3.9)$$

Then $P\chi_{[0,N]^d}$ satisfies the partition of unity condition (1.1) and belongs to $C^{2N-3}(\mathbb{R}^d)$.

Proof. The result follows almost immediately from (3.8); we just have to determine the value of a such that (2.6) is satisfied for $\mathbf{k} = \mathbf{0}$. Using that

$$\sin^{2N-2}(\pi x / N) = \left(\frac{e^{\pi i x / N} - e^{-\pi i x / N}}{2i} \right)^{2N-2} = \frac{1}{4^{N-1}} \sum_{k=-N+1}^{N-1} (-1)^k \binom{2N-2}{N-1+k} e^{2\pi i k x / N}$$

and pulling out the coefficient corresponding to $k = 0$, leads to the form in (3.9). \square

In Corollary 3.3 the regularity of $P\chi_{[0,N]^d}$ is related to the support size, i.e., to the parameter N . At the price of a more complicated construction, arbitrary (finite) regularity can be obtained even for $N = 2$.

Corollary 3.4 *Given $L \in \mathbb{N}$, let*

$$P(\mathbf{x}) = \prod_{j=1}^d \sin^{2L}(\pi x_j / 2) \sum_{k_j=0}^{L-1} \binom{2L-1}{k_j} \sin^{2(L-1-k_j)}(\pi x_j / 2) \cos^{2k_j}(\pi x_j / 2).$$

Then $P\chi_{[0,2]^d}$ satisfies the partition of unity condition and belongs to $C^{2L-1}(\mathbb{R}^d)$.

Proof. In the case $d = 1$, it was shown in [2] that letting

$$Q(x) := \sin^{2L}(\pi x/2) \sum_{k=0}^{L-1} \binom{2L-1}{k} \sin^{2(L-1-k)}(\pi x/2) \cos^{2k}(\pi x/2), \quad x \in \mathbb{R},$$

we have that $Q\chi_{[0,2]}$ belongs to $C^{2L-1}(\mathbb{R})$ and satisfies the partition of unity property. Since $P(\mathbf{x}) = \prod_{j=1}^d Q(x_j)$, we have

$$\begin{aligned} \sum_{\mathbf{n} \in \mathbb{Z}_N^d} P(\mathbf{x} + \mathbf{n}) &= \sum_{n_1=0}^{N-1} \cdots \sum_{n_d=0}^{N-1} Q(x_1 + n_1) \cdots Q(x_d + n_d) \\ &= \sum_{n_1=0}^{N-1} Q(x_1 + n_1) \cdots \sum_{n_d=0}^{N-1} Q(x_d + n_d) = 1. \end{aligned}$$

By construction, $P\chi_{[0,2]^d} \in C^{2L-1}(\mathbb{R}^d)$. This completes the proof. \square

4 Construction of Gabor frames in $L^2(\mathbb{R}^d)$

As application of our results in Section 2 and Section 3 we will now construct pairs of dual Gabor frames in $L^2(\mathbb{R}^d)$ with attractive properties. Other constructions in $L^2(\mathbb{R}^d)$ in the literature include [3] and [16]; we will comment on these along the way. Note also the approach (in $L^2(\mathbb{R})$) by Laugesen in [18]. In Section 5 we will show that the constructions presented here yield a very convenient way to obtain finite dual pairs of Gabor frames in $\ell^2(\mathbb{Z}^d)$ as well.

For $\mathbf{y} \in \mathbb{R}^d$, the translation operator $T_{\mathbf{y}}$ and the modulation operator $E_{\mathbf{y}}$, both acting on $L^2(\mathbb{R}^d)$, are defined by $(T_{\mathbf{y}}f)(\mathbf{x}) = f(\mathbf{x} - \mathbf{y})$, $(E_{\mathbf{y}}f)(\mathbf{x}) = e^{2\pi i \mathbf{y} \cdot \mathbf{x}} f(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^d$, where $\mathbf{y} \cdot \mathbf{x}$ denotes the inner product of \mathbf{y} and \mathbf{x} in \mathbb{R}^d . Given a real and invertible $d \times d$ matrix B and $g \in L^2(\mathbb{R}^d)$ we consider Gabor systems of the form

$$\{E_{B\mathbf{m}}T_{\mathbf{n}}g\}_{\mathbf{m}, \mathbf{n} \in \mathbb{Z}^d} = \{e^{2\pi i B\mathbf{m} \cdot \mathbf{x}} g(\mathbf{x} - \mathbf{n})\}_{\mathbf{m}, \mathbf{n} \in \mathbb{Z}^d}.$$

The theory for duality of Gabor systems is closely related to our discussion about partition of unities. Using the notation $B^\sharp = (B^T)^{-1}$, it is well known (see [20, 11, 17, 10]) that two Bessel sequences $\{E_{B\mathbf{m}}T_{\mathbf{n}}g\}_{\mathbf{m}, \mathbf{n} \in \mathbb{Z}^d}$ and $\{E_{B\mathbf{m}}T_{\mathbf{n}}h\}_{\mathbf{m}, \mathbf{n} \in \mathbb{Z}^d}$ form dual frames for $L^2(\mathbb{R}^d)$ if and only if

$$\sum_{\mathbf{k} \in \mathbb{Z}^d} \overline{g(\mathbf{x} - B^\sharp \mathbf{n} + \mathbf{k})} h(\mathbf{x} + \mathbf{k}) = |\det B| \delta_{\mathbf{n},0}, \quad \text{a.e. } \mathbf{x} \in \mathbb{R}^d, \quad (4.1)$$

for all $\mathbf{n} \in \mathbb{Z}^d$. We will now choose the functions g and h of the form

$$g(\mathbf{x}) = G(\mathbf{x})\chi_{[0,N]^d}(\mathbf{x}), \quad h(\mathbf{x}) = H(\mathbf{x})\chi_{[0,N]^d}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d,$$

for some $N \in \mathbb{N}$ and entire functions G, H . Then, for $\mathbf{n} = 0$, the condition (4.1) takes the form

$$\sum_{\mathbf{k} \in \mathbb{Z}^d} \overline{G(\mathbf{x} + \mathbf{k})} H(\mathbf{x} + \mathbf{k}) \chi_{[0, N]^d}(\mathbf{x} + \mathbf{k}) = |\det B|, \text{ a.e. } \mathbf{x} \in \mathbb{R}^d;$$

up to the factor $|\det B|$ this is clearly a partition of unity condition on the function $\overline{G}H\chi_{[0, N]^d}$. Thus, the results in Section 2 and Section 3 have almost immediate consequences for construction of dual Gabor frames. For the discussion of the regularity of the obtained constructions, we will use that if $P(\mathbf{x}) := \prod_{j=1}^d \sin^M(\pi x_j/N) A(\mathbf{x})$ for some real-valued and continuous $(N\mathbb{Z})^d$ -periodic trigonometric polynomial A , then $P\chi_{[0, N]^d} \in C^{M-1}(\mathbb{R}^d)$. The relation between the parameter M in the following Theorem 4.1 and the parameter L in Theorem 3.1 is that $L = 2M$.

Theorem 4.1 *Let $K, N, M \in \mathbb{N}$ with $M \leq K$. Assume that*

$$P(\mathbf{x}) = \sum_{\mathbf{k} \in (\mathbb{Z} \cap [-K, K])^d} c_{\mathbf{k}} e^{2\pi i \mathbf{k} \cdot \mathbf{x}/N}, \quad \mathbf{x} \in \mathbb{R}^d$$

is a real-valued trigonometric polynomial with $c_{\mathbf{k}} = \frac{1}{N^d} \delta_{\mathbf{k}, \mathbf{0}}$, $\forall \mathbf{k} \in (N\mathbb{Z})^d$, and that $P\chi_{[0, N]^d} \in C^{2M-1}(\mathbb{R}^d)$. Let P be factorized as

$$P(\mathbf{x}) = \prod_{j=1}^d \sin^{2M}(\pi x_j/N) G(\mathbf{x}) H(\mathbf{x}) \quad (4.2)$$

for some $(N\mathbb{Z})^d$ -periodic real-valued trigonometric polynomials G, H . Let B be a real and invertible $d \times d$ matrix such that

$$B^\sharp \mathbf{n} \notin (-N, N)^d, \quad \forall \mathbf{n} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}. \quad (4.3)$$

Then the functions

$$\begin{aligned} g(\mathbf{x}) &= \left(\prod_{j=1}^d \sin^M(\pi x_j/N) \right) G(\mathbf{x}) \chi_{[0, N]^d}(\mathbf{x}), \\ h(\mathbf{x}) &= |\det B| \left(\prod_{j=1}^d \sin^M(\pi x_j/N) \right) H(\mathbf{x}) \chi_{[0, N]^d}(\mathbf{x}) \end{aligned}$$

belong to $C^{M-1}(\mathbb{R}^d)$ and generate dual frames $\{E_{B\mathbf{m}} T_{\mathbf{n}} g\}_{\mathbf{m}, \mathbf{n} \in \mathbb{Z}^d}$ and $\{E_{B\mathbf{m}} T_{\mathbf{n}} h\}_{\mathbf{m}, \mathbf{n} \in \mathbb{Z}^d}$ for $L^2(\mathbb{R}^d)$.

Proof. Since $B^\sharp \mathbf{n} \notin (-N, N)^d$, $\forall \mathbf{n} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$, (4.1) is satisfied for $\mathbf{n} \neq \mathbf{0}$; thus the result follows from Theorem 3.1 and the comment just before the formulation of Theorem 4.1. \square

Some comments are in order:

- (i) Under the assumptions in Theorem 4.1, we know from Theorem 3.1 that factorizations of P as in (4.2) in terms of trigonometric polynomials G, H always exist. For example, we can choose $G(\mathbf{x}) = \prod_{j=1}^d e^{2\pi i x_j M/N}$ and $H(\mathbf{x}) = A_{2M}(\mathbf{x})$, where A_{2M} is defined as in (3.3).
- (ii) The construction in Theorem 4.1 is much simpler than the one given in [3]. We avoid a complicated book keeping; and we avoid to enlarge the support of the dual window and can keep the same support size as for the window itself. The construction by I. Kim in [15, 16] also provides dual windows with the same support as the given window, but without an explicit expression for the dual window.
- (iii) Small adjustments of Theorem 4.1 lead to constructions of tight frames. If we (in addition to the stated conditions) assume that the trigonometric polynomial P is non-negative, the function

$$k(\mathbf{x}) := \sqrt{|\det B|} \left(\prod_{j=1}^d \sin^M(\pi x_j/N) \right) \sqrt{G(\mathbf{x})H(\mathbf{x})} \chi_{[0,N]^d}(\mathbf{x}),$$

belongs to $C^{M-1}(\mathbb{R}^d)$ and generates a tight Gabor frame $\{E_{B\mathbf{m}}T_{\mathbf{n}}k\}_{\mathbf{m},\mathbf{n} \in \mathbb{Z}^d}$ with frame bound 1.

- (iv) A simple scaling extends Theorem 4.1 to a construction of dual frames $\{E_{B\mathbf{m}}T_{C\mathbf{n}}g\}_{\mathbf{m},\mathbf{n} \in \mathbb{Z}^d}$ and $\{E_{B\mathbf{m}}T_{C\mathbf{n}}h\}_{\mathbf{m},\mathbf{n} \in \mathbb{Z}^d}$, where B and C are real-valued invertible matrices. We leave the formulation of this result to the interested reader.

The following construction of a tight frame based on Corollary 3.3 and the bullet (iii) above appears to be particularly simple and useful.

Corollary 4.2 *Let $N \in \mathbb{N}$ with $N \geq 2$, and let B be a real and invertible $d \times d$ matrix such that (4.3) holds. Let*

$$k(\mathbf{x}) := \sqrt{|\det B|} \left(\frac{4^{N-1}}{N \binom{2N-2}{N-1}} \right)^{d/2} \prod_{j=1}^d \sin^{N-1}(\pi x_j/N) \chi_{[0,N]^d}(\mathbf{x}).$$

Then $k \in C^{N-2}(\mathbb{R}^d)$, and $\{E_{B\mathbf{m}}T_{\mathbf{n}}k\}_{\mathbf{m},\mathbf{n} \in \mathbb{Z}^d}$ is a tight Gabor frame with frame bound 1.

Note that this particular construction resembles the original work by Daubechies, Grossmann, and Meyer [6]: the difference is that [6] only deals with the case $N = 2$ and the dimension $d = 1$. It is clear that in Corollary 4.2 the possibility to increase N is the key to the higher regularity. We also note that unless the matrix B is a diagonal matrix, the construction in Corollary 4.2 is not a tensor product of the constructions in [6].

Arbitrary (finite) regularity can be obtained for the windows k in Corollary 4.2 by increasing the parameter $N \in \mathbb{N}$, but the price is that the support size increases as well. The following alternative construction is based on the condition (4.3) with the choice $N = 2$, but nevertheless it allows us to obtain arbitrary high regularity. The merits of this particular

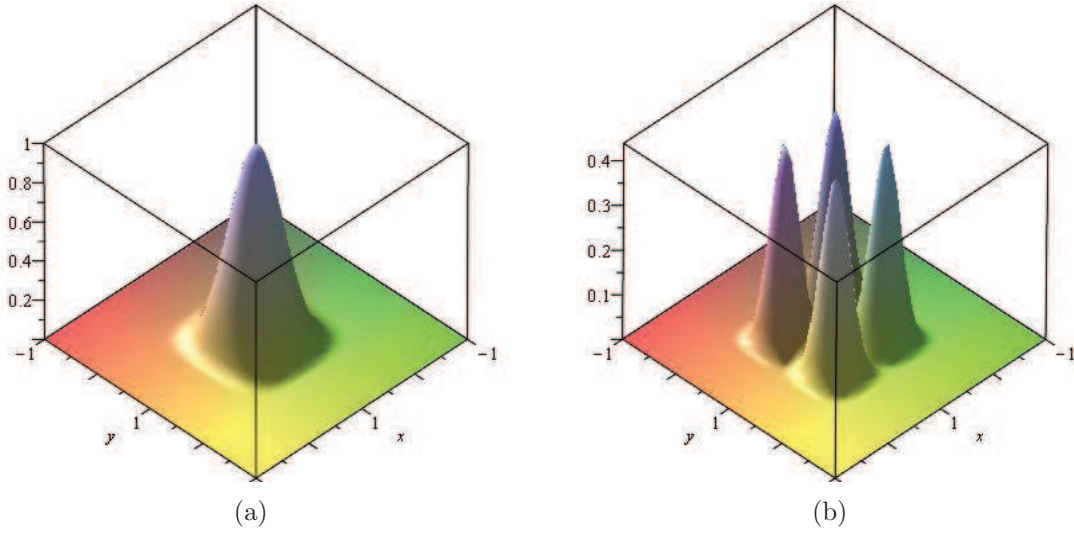


Figure 1: The functions g and h in Corollary 4.3 for $L = 3$, $N = d = 2$ and the matrix B with entries $b_{11} = b_{22} = 1/2$, $b_{12} = 0$, $b_{21} = -1/2$.

construction are twofold: we are able to increase the regularity while we keep the support $[0, 2]^d$, and the redundancy of the resulting Gabor frames is minimized among the ones satisfying (4.3) (the smallest values for $|\det B^\sharp| = |\det B|^{-1}$ for matrices B satisfying (4.3) are clearly obtained whenever $N = 2$.)

Corollary 4.3 *Let $L \in \mathbb{N}$, and let B be a real and invertible $d \times d$ matrix such that (4.3) holds with $N = 2$. Define*

$$g(\mathbf{x}) = \prod_{j=1}^d \sin^L(\pi x_j/2) \chi_{[0,2]^d}(\mathbf{x})$$

and

$$h(\mathbf{x}) = |\det B| \prod_{j=1}^d \sin^L(\pi x_j/2) \sum_{k_j=0}^{L-1} \binom{2L-1}{k_j} \sin^{2(L-1-k_j)}(\pi x_j/2) \cos^{2k_j}(\pi x_j/2) \chi_{[0,2]^d}(\mathbf{x}).$$

Then $g \in C^{L-1}(\mathbb{R}^d)$, $h \in C^{L-1}(\mathbb{R}^d)$, and the functions $\{E_{B\mathbf{m}}T_{\mathbf{n}}g\}_{\mathbf{m}, \mathbf{n} \in \mathbb{Z}^d}$ and $\{E_{B\mathbf{m}}T_{\mathbf{n}}h\}_{\mathbf{m}, \mathbf{n} \in \mathbb{Z}^d}$ form a pair of dual frames.

Proof. Using Corollary 3.4, it follows that the functions g and h satisfy the condition (4.1) for $n = 0$. The choice of B and the support sizes for g and h shows that (4.1) holds for $n \neq 0$ as well. \square

We illustrate Corollary 4.3 in Figure 1, which is based on the choices $L = 3$, $N = d = 2$ and $B = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$.

We will now give a closer analysis of the central condition (4.3). For any $d \times d$ matrix, define the norm $\|B\|$ by

$$\|B\| = \sup_{\|\mathbf{x}\|=1} \|B\mathbf{x}\|.$$

We first state the following simple sufficient condition for (4.3) to hold.

Lemma 4.4 *The condition (4.3) is satisfied if $\|B\| \leq \frac{1}{\sqrt{d}N}$.*

Proof. Since B is invertible, for any $\mathbf{n} \in \mathbb{Z}^d$ we have

$$\|\mathbf{n}\| = \|B^T B^\sharp \mathbf{n}\| \leq \|B\| \|B^\sharp \mathbf{n}\|;$$

thus, for $\mathbf{n} \neq 0$, $\|B^\sharp \mathbf{n}\| \geq \|\mathbf{n}\|/\|B\| \geq \sqrt{d}N$ if $\|B\| \leq 1/(\sqrt{d}N)$. Therefore, $B^\sharp \mathbf{n} \notin (-N, N)^d$ for $\mathbf{n} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$. \square

The following example shows that (at least for $N = 2$) the norm condition $\|B\| \leq 1/(\sqrt{d}N)$ is optimal, in the sense that we for any number $a > 1/(\sqrt{d}N)$ can find an invertible matrix B with $\|B\| = a$ such that (4.3) is not satisfied:

Example 4.5 Let $N = d = 2$. Then for any $\epsilon > 0$, there exists an invertible 2×2 matrix B_ϵ with $\|B_\epsilon\| = \frac{1+\epsilon}{2\sqrt{2}}$ such that $B_\epsilon^\sharp \mathbf{n} \in (-2, 2)^2$ for some $\mathbf{n} \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}$. In fact, consider $B_\epsilon = \frac{1+\epsilon}{4} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. Then we have

$$\begin{aligned} \|B_\epsilon\| &= \sup_{\theta} \left\| \frac{1+\epsilon}{4} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} \right\| \\ &= \sup_{\theta} \frac{1+\epsilon}{4} \sqrt{(\cos \theta + \sin \theta)^2 + (\cos \theta - \sin \theta)^2} = \frac{1+\epsilon}{2\sqrt{2}}. \end{aligned}$$

But $B_\epsilon^\sharp(1, 0)^T = \frac{2}{1+\epsilon} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{2}{1+\epsilon} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \in (-2, 2)^2$. Thus (4.3) does not hold. \square

Easy calculations show that in the special case of a real and invertible $d \times d$ diagonal matrix B , the condition (4.3) is satisfied if and only if $\|B\| \leq 1/N$. On the other hand, the following example shows that (4.3) can be satisfied for non-diagonal matrices B with arbitrary large norm. Also in the work by I. Kim [15] it was mentioned that Gabor frame constructions with large matrix norm $\|B\|$ are possible.

Example 4.6 Let $N = d = 2$. There exist invertible 2×2 matrices $B_a, a > 0$, such that $(B_a^\sharp \mathbf{n}) \notin (-2, 2)^2$ for all $\mathbf{n} \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}$ but $\lim_{a \rightarrow \infty} \|B_a\| = \infty$. In fact, consider $B_a = \frac{1}{4} \begin{pmatrix} 2 & 0 \\ -2a & 2 \end{pmatrix}$. Then $B_a^\sharp = \begin{pmatrix} 2 & 2a \\ 0 & 2 \end{pmatrix}$. Let $\mathbf{n} = (n_1, n_2)^T \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}$. Then $B_a^\sharp \mathbf{n} =$

$n_1(2, 0)^T + n_2(2a, 2)^T$. If $n_2 \neq 0$, then $(B_a^\sharp \mathbf{n})_2 \notin (-2, 2)$; if $n_2 = 0$ and $n_1 \neq 0$, then $(B_a^\sharp \mathbf{n})_1 \notin (-2, 2)$. Thus $(B_a^\sharp \mathbf{n}) \notin (-2, 2)^2$, $\forall \mathbf{n} \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}$. Also,

$$\|B_a\| = \sup_{\theta} \frac{1}{4} \sqrt{(2 \cos \theta)^2 + (-2a \cos \theta + 2 \sin \theta)^2} \geq \frac{1}{2}a.$$

Hence $\lim_{a \rightarrow \infty} \|B_a\| = \infty$. \square

Relating to the one-dimensional case, we know that if $\{E_{mb}T_n g\}_{m,n \in \mathbb{Z}}$ is a Gabor frame $L^2(\mathbb{R})$, then $0 < b \leq 1$. The corresponding statement in higher dimensions is that $|\det B| \leq 1$ is a necessary condition for $\{E_{B\mathbf{m}}T_{\mathbf{n}} g\}_{m,n \in \mathbb{Z}}$ to be a frame for $L^2(\mathbb{R}^d)$. In contrast to the one-dimensional case, the following example shows that (4.3) might not be satisfied for $d \geq 2$, regardless how small $|\det B|$ is.

Example 4.7 Fix $\epsilon > 0$ and consider for $a > 0$ the matrix $B_a = \epsilon^{1/2} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$. Then $\det B_a = \epsilon$ for all $a > 0$, and $\|B_a\| = \epsilon^{1/2} \max\{a, a^{-1}\}$. Also, $B_a^\sharp = \frac{1}{\epsilon^{1/2}} \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix}$. Clearly $B_a^\sharp \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\epsilon^{1/2}} \begin{pmatrix} a^{-1} \\ 0 \end{pmatrix}$, $B_a^\sharp \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\epsilon^{1/2}} \begin{pmatrix} 0 \\ a \end{pmatrix}$; thus, for an arbitrary value of ϵ we can find $a > 0$ such that $\det B_a = \epsilon$ but for some $\mathbf{n} \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}$, $B^\sharp \mathbf{n} \in (-2, 2)^2$. The example easily extends to any \mathbb{R}^d , $d \geq 2$. \square

5 Gabor frames in $\ell^2(\mathbb{Z}^d)$ through sampling

As further application of the results we will now show that the attractive properties of the dual pairs of Gabor frames in Section 4 yield an easy way to construct Gabor frames in $\ell^2(\mathbb{Z}^d)$ through sampling. For general functions in $L^2(\mathbb{R})$ the work by Janssen [12] shows that sampling is a delicate issue, but the continuity and compact support of the windows constructed in Section 4 remove several technical difficulties. For further information on discrete Gabor systems we refer to the paper [11] by Janssen (which also deals with the more general case of shift-invariant systems), [4, 5] by Cvetković and Vetterli, as well as to the recent paper [19] by Lopez and Han. We also mention that the theory for translation invariant systems on LCA groups yields a joint framework to Gabor theory on $L^2(\mathbb{R}^d)$ and $\ell^2(\mathbb{Z}^d)$, see, e.g., the papers [13, 14] by Jakobsen and Lemvig.

Let B denote an invertible $d \times d$ matrix for which B^{-1} has integer entries. Consider the subgroup $G := B^{-1}\mathbb{Z}^d$ of \mathbb{Z}^d , and let Ω denote a collection of coset representatives of the coset \mathbb{Z}^d/G ; that is, \mathbb{Z}^d is a disjoint union of the sets $G + \mathbf{m}$, where $\mathbf{m} \in \Omega$. It is well known that the number of elements in Ω is

$$|\Omega| = |\det(B^{-1})| = \frac{1}{|\det B|}.$$

Let us now fix any sequence in $\ell^2(\mathbb{Z}^d)$; for our current purpose it will be convenient to denote the sequence by $\{c(\mathbf{j})\}_{\mathbf{j} \in \mathbb{Z}^d}$ rather than $\{c_{\mathbf{j}}\}_{\mathbf{j} \in \mathbb{Z}^d}$. We will consider the *Gabor system* in $\ell^2(\mathbb{Z}^d)$ generated by the sequence $\{c(\mathbf{j})\}_{\mathbf{j} \in \mathbb{Z}^d}$ and the matrix B , i.e., the collection of sequences $\{c_{\mathbf{m},\mathbf{n}}\}_{\mathbf{m},\mathbf{n} \in \mathbb{Z}^d} \subset \ell^2(\mathbb{Z}^d)$ given by

$$c_{\mathbf{m},\mathbf{n}}(\mathbf{j}) = e^{2\pi i B \mathbf{m} \cdot \mathbf{j}} c(\mathbf{j} - \mathbf{n}), \mathbf{j} \in \mathbb{Z}^d.$$

Given two sequences $\{c(\mathbf{j})\}_{\mathbf{j} \in \mathbb{Z}^d}, \{d(\mathbf{j})\}_{\mathbf{j} \in \mathbb{Z}^d} \in \ell^2(\mathbb{Z}^d)$ such that $\{c_{\mathbf{m},\mathbf{n}}\}_{\mathbf{m},\mathbf{n} \in \mathbb{Z}^d}$ and $\{d_{\mathbf{m},\mathbf{n}}\}_{\mathbf{m},\mathbf{n} \in \mathbb{Z}^d}$ are Bessel sequences, it was shown in Theorem 1.4 in [19] that $\{c_{\mathbf{m},\mathbf{n}}\}_{\mathbf{m},\mathbf{n} \in \mathbb{Z}^d}$ and $\{d_{\mathbf{m},\mathbf{n}}\}_{\mathbf{m},\mathbf{n} \in \mathbb{Z}^d}$ are dual frames for $\ell^2(\mathbb{Z}^d)$ if and only if

$$\sum_{\mathbf{k} \in \mathbb{Z}^d} \overline{c(\mathbf{j} - B^\sharp \mathbf{n} + \mathbf{k})} d(\mathbf{j} + \mathbf{k}) = |\det B| \delta_{\mathbf{n},0} \quad (5.1)$$

for all $\mathbf{j}, \mathbf{n} \in \mathbb{Z}^d$.

The relation to the results in Section 4 is evident. In fact, for all the frame constructions in Section 4, the windows $g(x) = G(x)\chi_{[0,N]}(x)$ and the dual windows $h(x) = H(x)\chi_{[0,N]}(x)$ are continuous functions with compact support, which implies that the duality conditions (4.1) hold *pointwise* for all $x \in \mathbb{R}^d$; this clearly implies that the sequences $\{c(\mathbf{j})\}_{\mathbf{j} \in \mathbb{Z}^d} := \{g(\mathbf{j})\}_{\mathbf{j} \in \mathbb{Z}^d}$ and $\{d(\mathbf{j})\}_{\mathbf{j} \in \mathbb{Z}^d} := \{h(\mathbf{j})\}_{\mathbf{j} \in \mathbb{Z}^d}$ satisfy (5.1). In other words: the samples of the dual windows g, h , i.e., the sequences $\{g(\mathbf{j})\}_{\mathbf{j} \in \mathbb{Z}^d}$ and $\{h(\mathbf{j})\}_{\mathbf{j} \in \mathbb{Z}^d}$ generate dual Gabor frames for $\ell^2(\mathbb{Z}^d)$. Thus, Theorem 4.1 has the following immediate consequence:

Corollary 5.1 *Under the assumptions in Theorem 4.1, the sequences*

$$\begin{aligned} g(\mathbf{j}) &:= \left(\prod_{\ell=1}^d \sin^M(\pi \mathbf{j}_\ell / N) \right) G(\mathbf{j}) \chi_{[0,N]^d}(\mathbf{j}), \mathbf{j} \in \mathbb{Z}^d, \\ h(\mathbf{j}) &= |\det B| \left(\prod_{\ell=1}^d \sin^M(\pi \mathbf{j}_\ell / N) \right) H(\mathbf{j}) \chi_{[0,N]^d}(\mathbf{j}), \mathbf{j} \in \mathbb{Z}^d, \end{aligned}$$

belong to $\ell^2(\mathbb{Z}^d)$ and the associated discrete Gabor systems $\{g_{\mathbf{m},\mathbf{n}}\}_{\mathbf{m},\mathbf{n} \in \mathbb{Z}^d}$ and $\{h_{\mathbf{m},\mathbf{n}}\}_{\mathbf{m},\mathbf{n} \in \mathbb{Z}^d}$ form dual frames for $\ell^2(\mathbb{Z}^d)$.

Similarly, Corollary 4.2 leads to the following explicit constructions of finite tight Gabor frames in $\ell^2(\mathbb{Z}^d)$:

Corollary 5.2 *Let $N \in \mathbb{N}$ with $N \geq 2$, and let B be a real and invertible $d \times d$ matrix for which B^{-1} has integer entries and (4.3) holds. Let*

$$k(\mathbf{j}) := \sqrt{|\det B|} \left(\frac{4^{N-1}}{N \binom{2N-2}{N-1}} \right)^{d/2} \prod_{\ell=1}^d \sin^{N-1}(\pi \mathbf{j}_\ell / N) \chi_{[0,N]^d}(\mathbf{j}), \mathbf{j} \in \mathbb{Z}^d.$$

Then $\{k(\mathbf{j})\}_{\mathbf{j} \in \mathbb{Z}^d} \in \ell^2(\mathbb{Z}^d)$, and the associated discrete Gabor system $\{k_{\mathbf{m},\mathbf{n}}\}_{\mathbf{m},\mathbf{n} \in \mathbb{Z}^d}$ is a tight Gabor frame for $\ell^2(\mathbb{Z}^d)$ with frame bound 1.

Note that sampling of the frame in Corollary 4.3 yields the trivial dual pair of Gabor frames in $\ell^2(\mathbb{Z}^d)$, with windows vanishing at all other points than $(1, 1, \dots, 1)$.

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